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Rayleigh type bending waves in anisotropic media

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Abstract

The existence, dispersion properties, velocities and energy of waves, localized near the stress-free edge of thin anisotropic plates are investigated. As shown, some qualitatively new effects occur: the velocity of Rayleigh type waves can be not minimal between bending waves; wave decay takes place with oscillations; under some type of anisotropy, power flow can equal zero and can change the sign. The well-known Leontovich–Lighthill theorem does not hold any longer, despite the same sign of the phase and group velocities the power flow can have the same or opposite sign.

1. Introduction

Bending waves of Rayleigh type (BWRT) were treated first by Konenkov (1960) for isotropic media [1], where the existence of waves were proven together with the analysis of their properties. Since this paper, such waves have been investigated by many authors (see, for example, Refs. [2–6]). But in contrast to the usual in-plane Rayleigh waves [7] which enable some non-destructive testing, seismic monitoring, etc. interest was not very high due to the relatively small decay factor (of the order v^4, where v is the Poisson ratio) for isotropic media. So, the main interest was focused on a mathematical formalism. Further progress in this field concerned two directions. The first devoted to the waves at the "interface" (edge-by-edge contact between plates [8,9]), which represent an analogue of the Stonely waves at bending. Second dealt with the edge waves in plates, immersed in fluid, of interest for the applications in hydroelasticity and mechanics of ice [10–12]. Some papers recorded results which overlapped (see, e.g., numerous comments in Ref. [13]) and have been rediscovered many times.

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However, with the wide spread of advanced composite materials, many of which are highly anisotropic, certain theoretical and practical questions are pertinent:

1. Do BWRT exist in media with general anisotropy?
2. If they exist, what are their properties (energy, factor of the exponential decay, etc.)?
3. What is the influence of layup in laminates on the wave properties (i.e., symmetrical layup or asymmetrical layup with coupled bending and stretching [14])?

Despite some interest, shown in Refs. [5,6] for the orthotropic materials with simplest orientation, the detailed investigation on the subject has not been published. In the present paper two of the mentioned questions are considered and answered.

2. Physical and mathematical statement of problem

Consider a thin laminate with symmetrical layup, made of perfectly joined anisotropic plies. The total thickness of laminate is $2h$ and Cartesian co-ordinates x_1, x_2, and $x_3 = z$ are normalized over h. The laminate geometry is shown in Fig. 1. The internal stress–strain state (SSS) of the laminate is supposed to be long-wave, i.e., it satisfies the classical relations of the 2-D anisotropic plate bending theory [15]. In what follows the dimensionless quantities are considered, namely the elastic moduli are normalized over the maximal Young's modulus of the plies, and the mass densities are normalized in a similar manner over the maximal mass density of the plies.

The main relations for the normal deflection w, slopes θ_α, longitudinal displacements u_α, torques $M_{\alpha\beta}$, and transversal forces $Q_{\alpha z}$ are

$$\theta_\alpha = -\partial_\alpha w, \quad u_\alpha = z\theta_\alpha \quad (\alpha = 1, 2),$$
$$M_{11} = -(d_{11}^3\partial_1^2 + d_{16}^3\partial_1\partial_2 + d_{12}^3\partial_2^2)w, \quad M_{12} = -(d_{16}^3\partial_1^2 + d_{66}^3\partial_1\partial_2 + d_{62}^3\partial_2^2)w,$$
$$Q_{\alpha z} = \partial_1 M_{1\alpha} + \partial_2 M_{2\alpha} \quad (1 \leftrightarrow 2),$$

where $\mathbf{D} = \left\|d_{pq}^3\right\|$ is a matrix of bending stiffness. The normal deflection also satisfies the equation of motion

$$\partial_\alpha Q_{\alpha z} = \rho\partial_t^2 w,$$

where t is time and ρ is dimensionless integral mass density.

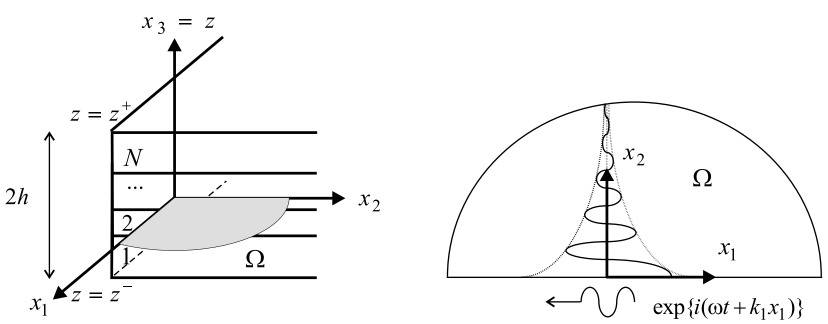

Fig. 1. Laminate and layup (left) and BWRT in the co-ordinate half-plane (right).

The harmonic oscillations of a semi-infinite laminate which occupies a region are investigated which occupies a region $\Omega : x_2 \geqslant 0, -\infty < x_1 < \infty$ in its plane. The edge of the laminate $x_2 = 0$ is assumed to be free of external loading.

Remark 1. In this paper the term *stress-free edge* is used in accordance with the accepted concept for thin plates, whereby the main SSS is subdivided into the internal SSS and the boundary layer (BL), not considered here. So, the internal SSS is described by the classical Kirchhoff's model of thin plate with respective integral boundary conditions, and all corrections are included in BL. Thus, Kirchhoff's theory is valid at the distance of a few h near the edge.

Then the normal bending $w = w^*(x_1, x_2)e^{i\omega t}$, normalized over h, satisfies the equation of oscillation with frequency ω, by

$$\{L_3(\partial_1, \partial_2) - \rho\omega^2\} w^* = 0, \tag{2.1}$$

$$L_3(\partial_1, \partial_2) \equiv d_{11}^3 \partial_1^4 + 4d_{16}^3 \partial_1^3 \partial_2 + 2(d_{12}^3 + 2d_{66}^3)\partial_1^2\partial_2^2 + 4d_{26}^3 \partial_1 \partial_2^3 + d_{22}^3 \partial_2^4.$$

At the edge, the boundary conditions are satisfied by

$$M(\partial_1, \partial_2) w^* = 0, \quad F(\partial_1, \partial_2) w^* = 0, \tag{2.2}$$

$$\begin{bmatrix} M \\ F \end{bmatrix} = -\begin{bmatrix} d_{12}^3 \partial_1^2 + d_{26}^3 \partial_1 \partial_2 + d_{22}^3 \partial_2^2 \\ 2d_{16}^3 \partial_1^3 + (d_{12}^3 + 4d_{66}^3)\partial_1^2 \partial_2 + 4d_{26}^3 \partial_1 \partial_2^2 + d_{22}^3 \partial_2^3 \end{bmatrix},$$

where M and F are operators responsible for the normal moment M_{22} and for the transversal Kirchhoff's shear force $P_{2z} = 2\partial_1 M_{12} + \partial_2 M_{22}$.

In investigating the existence of solutions propagating along the edge when distant from the edge inside the laminate, i.e., the desired BWRT

$$w^* = Ae^{i(k_1 x_1 + k_2 x_2)}, \quad A = Const, \quad \operatorname{Im} k_2 < 0.$$

Since $x_2 \geqslant 0$ the latter inequality provides the exponential decay along the half-axis $x_2 \geqslant 0$ (see Fig. 1).

Let

$$d_{pq} = \frac{d_{pq}^3}{d}, \quad s^4 = \frac{\rho\omega^2}{dk_1^4}, \quad \xi = \frac{k_2}{k_1},$$

where d is a particular bending stiffness, chosen for normalization (for example, a maximal one). For the definitiveness, set $k_1 > 0$. Substitution of w^* into Eq. (2.1) yields the characteristic equation with constant coefficients for the variable ξ

$$L(1, \xi) - s^4 \equiv d_{11} + 4d_{16}\xi + 2(d_{12} + 2d_{66})\xi^2 + 4d_{26}\xi^3 + d_{22}\xi^4 - s^4 = 0. \tag{2.3}$$

When using the normalized coefficients d_{pq} for the operator symbol from Eq. (2.1) another notation $L(1, \xi)$ is introduced. The roots of Eq. (2.3) describe all types of monochromatic waves and only conditions (2.2) should be satisfied additionally. The following propositions hold.

Proposition 1. *When the twist coupling stiffnesses are absent* ($d_{16} = d_{26} = 0$), Eq. (2.3) *can have pure imaginary roots* $\xi : \operatorname{Re}\xi = 0$; *in the contrary case, roots are real or complex.*

Under presence of the twist coupling stiffnesses Eq. (2.3) immediately leads to a contradiction since for the pure imaginary ξ the real and imaginary part of the left side must equal zero.

Proposition 2. *All complex roots of* Eq. (2.3) *are conjugated; two pairs of complex conjugated roots (and respective by the BWRT) can exist only at*

$$s < s^*, \quad s^{*4} = \inf_{\xi \in R} L(1, \xi).$$

In fact, the characteristic polynomial of the fourth order $L(1, \xi)$ is positively determined for real values ξ [14]. Since this polynomial is a smooth function, Eq. (2.3) at $s = s^*$ has at least one real root of multiplicity 2. At $s > s^*$ the number of real roots is not less than 2. Thus, at $s \geqslant s^*$ not more than one pair of complex conjugated roots exists, and this is insufficient to satisfy two boundary conditions (2.2) with simultaneous exponential decay along the axis x_2.

Proposition 3. *The BWRT phase velocity $V_R = -\omega/k_1$ has the upper bound*

$$|V_R| < V^*, \quad V^* = s^{*2} k_1 \sqrt{d/\rho}.$$

This is a simple physical corollary of Proposition 2.

Remark 2. It is noticeable that the natural direction of propagation for the chosen wave is against the direction of axis x_1.

The desired pair of complex roots is denoted as $\xi_{1,2}(\operatorname{Im} \xi_{1,2} > 0)$. Boundary conditions (2.2) acquire the form

$$\det \mathbf{\Delta}(s) = 0, \tag{2.4}$$

$$\mathbf{\Delta}(s) = \begin{bmatrix} d_{12} + 2d_{26}\xi_1 + d_{22}\xi_1^2 & d_{12} + 2d_{26}\xi_2 + d_{22}\xi_2^2 \\ 2d_{16} + (d_{12} + 4d_{66})\xi_1 + 4d_{26}\xi_1^2 + d_{22}\xi_1^3 & 2d_{16} + (d_{12} + 4d_{66})\xi_2 + 4d_{26}\xi_2^2 + d_{22}\xi_2^3 \end{bmatrix},$$

$$\frac{A_2}{A_1} = -\frac{d_{12} + 2d_{26}\xi_1 + d_{22}\xi_1^2}{d_{12} + 2d_{26}\xi_2 + d_{22}\xi_2^2}, \quad w^*(x_1, x_2) = \{A_1 \mathrm{e}^{\mathrm{i}\xi_1 k_1 x_2} + A_2 \mathrm{e}^{\mathrm{i}\xi_2 k_1 x_2}\} \mathrm{e}^{\mathrm{i}k_1 x_1} \tag{2.5}$$

Finally, the question of the existence of BWRT is reduced to the investigation of the roots s of Eq. (2.4) at the branches $\xi_1(s), \xi_2(s)$.

3. Case of orthotropic media

In the particular case of an orthotropic medium, whose principal axes coincide with the axes x_1, x_2, the situation is essentially simplified. Then the twist coupling stiffnesses $d_{16} = d_{26} = 0$ and the characteristic Eqs. (2.3) and (2.4) have only pure imaginary roots $\xi_1(s), \xi_2(s)$. On choosing the coefficient $d = d_{22}$, from Eqs. (2.3) and (2.4) one obtains the relations

$$\xi_{1,2} = \mathrm{i}\left\{C \mp \sqrt{D + s^4}\right\}^{1/2}, \quad D = C^2 - \frac{d_{11}}{d_{22}},$$
$$C = d_{12} + \frac{2d_{66}}{d_{22}}, \quad E = \frac{2d_{66}}{d_{22}}, \tag{3.1}$$

$$f(s) \equiv \left\{\frac{E + \sqrt{D + s^4}}{E - \sqrt{D + s^4}}\right\}^2 \left\{\frac{C - \sqrt{D + s^4}}{C + \sqrt{D + s^4}}\right\}^{1/2} = 1 \quad (f(s) = 1 \Leftrightarrow \det \mathbf{\Delta}(s) = 0). \tag{3.2}$$

At $s^4 \in \left[E^2 - D, C^2 - D\right]$ the function $f(s)$ varies from 0 to $+\infty$, i.e., the real root s of Eq. (3.2) exists and equals

$$s = \left\{-D + CE\left(2 - 3a^2 + 2\sqrt{2\left(a^2 - \frac{1}{2}\right)^2 + \frac{1}{2}}\right)\right\}^{1/4}, \quad a^2 = \frac{E}{C}, \tag{3.3}$$

where the positive definitiveness of the radicals follows from the positive definitiveness of the tensor of elastic constants. The magnitude ratio (2.5) is evidently positive. The physical quantities are given as the real part of the solution $\mathrm{Re}\{w^*(x_1, x_2)\mathrm{e}^{\mathrm{i}\omega t}\}$, and for the slopes θ_1, θ_2 and longitudinal displacements u_1, u_2 leading to

$$\theta_\alpha = -\partial_\alpha \mathrm{Re}\{w^*(x_1, x_2)\mathrm{e}^{\mathrm{i}\omega t}\} \quad (\alpha = 1, 2),$$

$$u_1 = -z\,\mathrm{Re}\left\{\mathrm{i}k_1 A_1\left[\mathrm{e}^{-k_1|\xi_1|x_2} + \frac{A_2}{A_1}\mathrm{e}^{-k_1|\xi_2|x_2}\right]\mathrm{e}^{\mathrm{i}(\omega t + k_1 x_1)}\right\},$$

$$u_2 = z\,\mathrm{Re}\left\{k_1 A_1\left[|\xi_1|\mathrm{e}^{-k_1|\xi_1|x_2} + |\xi_2|\frac{A_2}{A_1}\mathrm{e}^{-k_1|\xi_2|x_2}\right]\mathrm{e}^{\mathrm{i}(\omega t + k_1 x_1)}\right\}.$$

Thus, for a real magnitude A_1 the displacements u_1, u_2 (and the slopes θ_1, θ_2) are harmonic functions with the phase difference $-\pi/2$. During a full period the trajectory of an arbitrarily chosen point (x_1, x_2) is an ellipse, whose direction is counterclockwise and the semi-axes decay exponentially when at distance from the edge.

In the particular case of an isotropic (and of a transversely isotropic) medium equality (3.3) leads to the same relation, as found by Konenkov [1]

$$s = \{(1 - v)(3v - 1 + 2\sqrt{1 - 2v + 2v^2})\}^{1/4},$$

where v is the Poisson ratio.

It is also interesting to examine the qualitative behavior of the phase velocity V_R and its ratio over the velocity V_B of the ordinary bending wave. Consider the wave vector $\mathbf{k}$ with the angle of inclination φ with respect to the axis x_1. After replacing ∂_1 and ∂_2 in Eq. (2.1) by $|\mathbf{k}|\cos\varphi$ and $|\mathbf{k}|\sin\varphi$, respectively, the velocities V_B, V_R and their ratio $r(\varphi)$ are given by the formulas

$$V_B \equiv \frac{\omega}{|\mathbf{k}|} = \left[\frac{\omega^2 L_3(\cos\varphi, \sin\varphi)}{\rho}\right]^{1/4}, \quad V_R = -\frac{\omega}{k_1} = -s\left[\frac{\omega^2 d_{22}^3}{\rho}\right]^{1/4},$$

$$r(\varphi) = \frac{|V_R|}{V_B} = s\left\{\frac{d_{22}^3}{L_3(\cos\varphi, \sin\varphi)}\right\}^{1/4}. \tag{3.4}$$

In the case of isotropic materials the ratio $r(\varphi)$ is always less than unit [1].

Another important characteristic is the value of the averaged power flow $\mathfrak{I}$ across the section of a plate, normal to the direction of propagation x_1. As usual it is introduced using the integral of the product of stresses σ_{pq} and speeds $u_\alpha^\bullet$, $w^\bullet$, averaged over the period of oscillations [16]. Due to the definition, for the complex form of physical quantities

$$\mathfrak{I} = -\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t \int_0^{+\infty} \mathrm{d}x_2 \int_{-h}^{h} \left\{\operatorname{Re} u_1^\bullet \operatorname{Re}\sigma_{11} + \operatorname{Re} u_2^\bullet \operatorname{Re}\sigma_{12} + \operatorname{Re} w^\bullet \operatorname{Re}\sigma_{1z}\right\} \mathrm{d}z,$$

where the dot denotes the derivative with respect to time. After integrating over the thickness $\mathfrak{I}$ it acquires the final form

$$\mathfrak{I} = -\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t \int_0^{+\infty} \{\operatorname{Re}\theta_1^\bullet \operatorname{Re} M_{11} + \operatorname{Re}\theta_2^\bullet \operatorname{Re} M_{12} + \operatorname{Re} w^\bullet \operatorname{Re} Q_{1z}\} \, \mathrm{d}x_2. \tag{3.5}$$

4. Numerical example

For numerical illustration two types of orthotropic materials have been chosen: material T300/epoxy (T) with Young's moduli $E_1 = 130\,000, E_2 = 9750$, shear modulus $G_{12} = 6000\,\mathrm{N/mm^2}$, the Poisson ratio $\nu_{12} = 0.27$ and mass density $\rho = 1.58\,\mathrm{g/cm^3}$; material E-glass (E) with constants $\rho = 2$, $E_1 = 45\,000, E_2 = 13\,000, G_{12} = 4400, \nu_{12} = 0.29$. The principal axes coincide with the coordinate axes ($T/0$) or are rotated about the angle $\pi/2$ ($T/90$); the thickness is $2h = 1\,\mathrm{mm}$, coefficient $d = \max(d_{11}, d_{22})$.

The characteristic values s and the normalized power flow of BWRT are

$$s \approx 0.9983\,(T/0), 0.7323\,(T/90), 0.9920\,(E/0), 0.5234\,(E/90),$$

$$\min(\operatorname{Im}\xi_1, \operatorname{Im}\xi_2) \approx 0.01743\,(T/0), 0.004483\,(T/90), 0.04103\,(E/0), 0.02254\,(E/90),$$

$$\frac{\mathfrak{I}}{\omega d|k_1 A_1|^2} \approx -1.341(E/0), -0.3715\,(T/90), -1.734\,(E/0), -0.9601\,(E/90).$$

The plots of the resultant velocity ratio $r(\varphi)$ are shown in Fig. 2. As seen for different materials $r(\varphi)$ can be smaller or *greater* than unit. Thus, for the standard orientation of orthotropic materials one concludes that

1. the BWRT exists;
2. the decay factor $\min(\operatorname{Im}\xi_1, \operatorname{Im}\xi_2)$ can be higher than for an isotropic medium, where this value is no more than 10^{-2};

3. in contrast to the case of isotropic medium [1] the phase velocity of BWRT is no longer minimal between all possible bending waves (see Fig. 2).

5. General anisotropy

In the case of general anisotropy ($d_{16}, d_{26} \neq 0$) the analytical formulas for the roots $\xi_1(s), \xi_2(s)$ and characteristic value s are no longer available and to be found numerically. The procedure is to set the parameter s and from Eq. (2.3) two branches $\xi_1(s), \xi_2(s)$ are calculated and substituted into Eq. (2.4) (real and imaginary part of det $\mathbf{\Delta}(s)$). The results are illustrated for materials (T) and (E), whose principal axes x_1', x_2' are obtained by a rotation of the axes x_1, x_2 about the angle $0 < \psi < \pi/2$. Plots of the functions $s(\psi)$ are shown in Fig. 3. The respective branches $\xi_1(\psi), \xi_2(\psi)$ are presented in Figs. 4 and 5. The magnitude ratio (2.5) is shown in Fig. 6 and confirms that none of components can be neglected and the decay factor is defined by Re $\xi_1(\psi)$. This decay factor is shown in Figs. 4 and 5 by solid curves 1. As seen for (T) and (E) materials the exponential decay

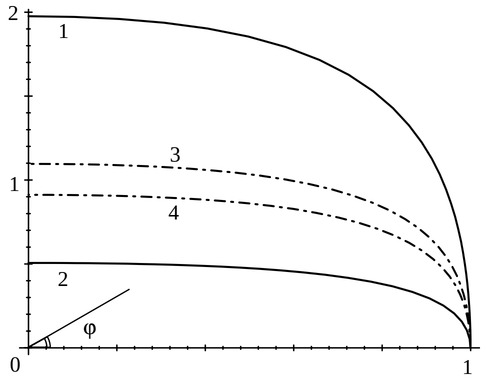

Fig. 2. $r(\varphi)$ plots in polar co-ordinates for materials T/0 (curve 1), T/90 (2), E/0 (3) and E/90 (4).

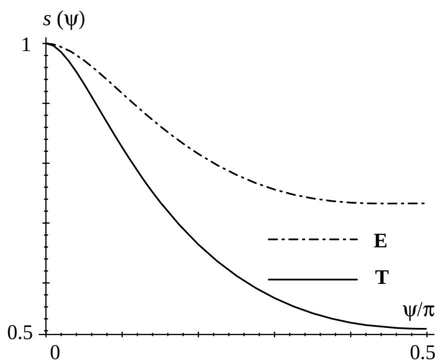

Fig. 3. Curves of frequency parameter for T (—) and E (–·–·) materials.

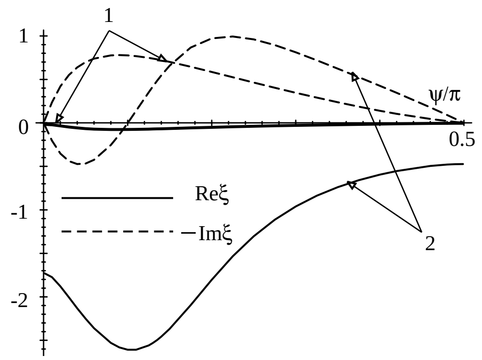

Fig. 4. Real (—) and imaginary (– – –) solutions for roots $\xi_1(\psi), \xi_2(\psi)$ for T-materials.

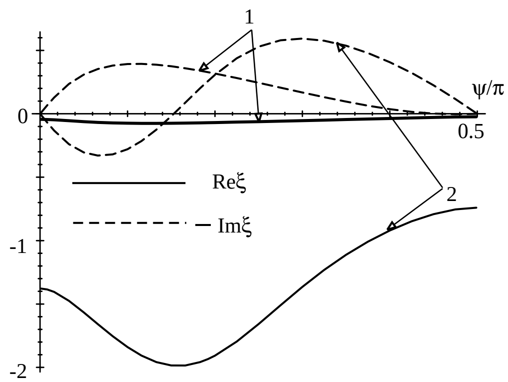

Fig. 5. Real (—) and imaginary (– – –) solutions for roots $\xi_1(\psi), \xi_2(\psi)$ for material E.

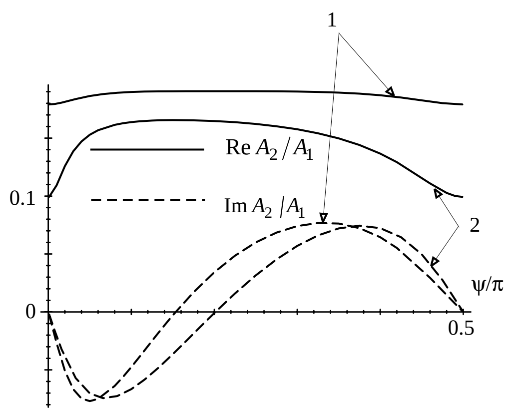

Fig. 6. The real (—) and imaginary (– – –) magnitude ratio A_2/A_1 as a function of ψ for T-materials (index 1) and E (index 2).

factor can acquire the values of the order 10^{-1} (at $\psi \approx 0.11\pi(T)$ and $\psi \approx 0.162\pi(E)$, respectively), which are ten times more that for the isotropic case [1].

6. Average power flow

The behavior of the power flow $\Im(\psi)$ is of special interest and plots of this normalized function are shown in Fig. 7. For $k_1 > 0$ the power flow seems to be negative. However, for both materials there exists a critical value $\psi^* = 0.1249\pi(T), 0.1767\pi(E)$ of the orientation angle for which the desired wave becomes steady, and then ($\psi > \psi^*$) it changes the direction of energy transfer. At the next critical angle $\psi^{**} = 0.1848\pi(T)$, $0.2268\pi(E)$ the wave becomes steady again and under $\psi > \psi^{**}$ the sign of the power flow is the same as the initial one. This fact is *new* and observed only for a medium with *general anisotropy* ($d_{16}, d_{26} \neq 0$). For isotropic media and for orthotropic media with standard orientation such a fact cannot be realized *in principle*.

To clarify the effect consider the generalized acoustical impedances I_m and I_p introduced as follows:

$$I_m^* = \frac{M_{11}}{\theta_1^\bullet} = -\frac{I_m}{V_R}, \quad I_p^* = \frac{P_{1z}}{w^\bullet} = -\frac{k_1^2 I_p}{V_R}, \quad I = I_m + I_p,$$

$$\theta_1 = -\mathrm{i}k_1 w, \quad \theta_1^\bullet = \omega k_1 w, \quad w^\bullet = \mathrm{i}\omega w.$$

After integration by parts the right side of Eq. (3.5) leads to the formulae

$$\Im = \Im_0 + \mathrm{Re}\{\mathrm{Re}\, w^\bullet\, \mathrm{Re}\, M_{12}\}|_{x_2=0}, \tag{6.1}$$

$$\begin{aligned}\Im_0 &\equiv -\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t \int_0^{+\infty} \{\mathrm{Re}\,\theta_1^\bullet\, \mathrm{Re}\, M_{11} + \mathrm{Re}\, w^\bullet\, \mathrm{Re}\, P_{1z}\}\, \mathrm{d}x_2 \\ &= \frac{\omega^2 k_1^2}{2V_R}\int_0^{+\infty} |w|^2 \mathrm{Re}(I_m + I_p)\, \mathrm{d}x_2.\end{aligned}$$

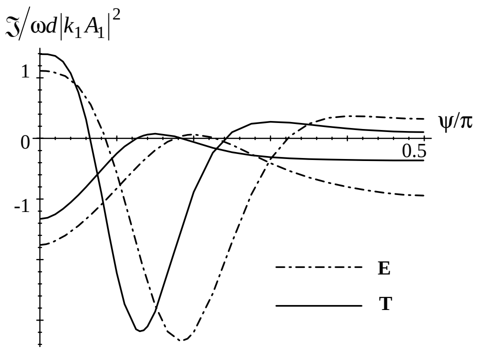

Fig. 7. Normalized averaged power flow and acoustic impedances $\mathrm{Re}(I_m + I_p)$ at $x_2 = 0$ (the latter are positive in the origin) for T-materials (—) and E-materials (–·–·).

Remark 3. The way to introduce the generalized impedances is not unique here. The chosen style deals with such quantities as bending moment and Kirchhoff's force, which directly participate in the formulation of boundary conditions. It also allows one to calculate to the magnitudes of the normal deflection for $\Im_0$.

The main contribution into power flow $\Im$ gives the variable $\Im_0$. Due to the simplest analogue for the harmonic oscillator with dissipation energy, subjected to the action of any external force F, consider a point of mass m on the spring with stiffness c under linear viscous friction with coefficient b. Its generalized acoustical impedance is

$$I^* \equiv \frac{F}{y^\bullet} = \frac{my^{\bullet\bullet} + by^\bullet + cy}{y^\bullet} = \mathrm{i}\left(m\omega - \frac{c}{\omega}\right) + b \tag{6.2}$$

and one can expect a positive value of $\operatorname{Re} I^*$ and the possible change of the sign of the imaginary part. So, for BWRT the most natural conclusion would be the constant sign of $\operatorname{Re}(I_m + I_p)$ and possible sign changing of $\operatorname{Im}(I_m + I_p)$.

Plots of the real and of the imaginary parts of the generalized impedances $I_m, I_p, I_m + I_p$ are shown in Figs. 8–10 for three values of the angle ψ: outside the interval $\left[\psi^*, \psi^{**}\right]$ (Figs. 8 and 10) and inside this interval (Fig. 9). As seen, outside the interval $\left[\psi^*, \psi^{**}\right]$ the analogue with Eq. (6.2)

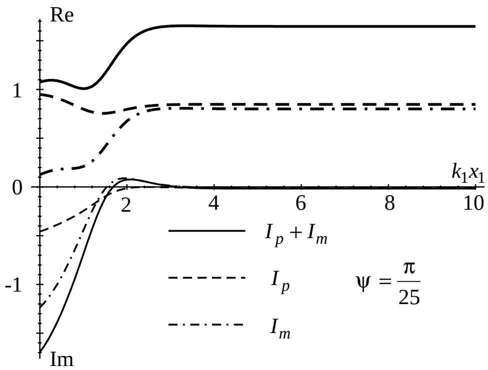

Fig. 8. Normalized impedances for $\psi < \psi^*(T)$ ($\varphi = \pi/25$). Key: —, $I_p + I_m$;−−−, I_p; −·−, I_m.

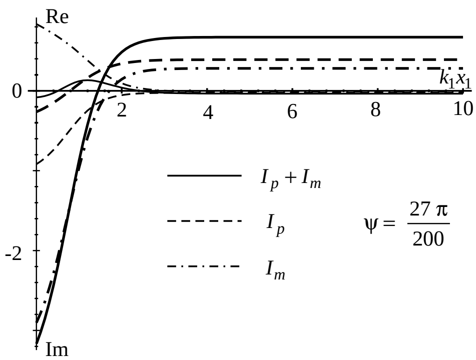

Fig. 9. Normalized impedances for $\psi^* < \psi < \psi^{**}$ ($\psi = 27\pi/200$). Key as for Fig. 8.

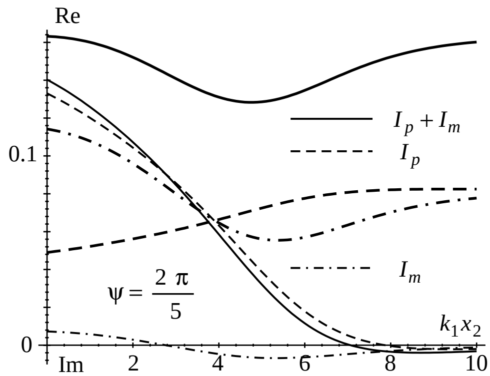

Fig. 10. Normalized impedances for $\psi > \psi^*$ ($\psi = 2\pi/5$). Key as for Fig. 8.

holds, but for the intermediate value of ψ one can single out an active zone near the edge with opposite direction of energy flow. This is a physical reason of the sign changing of the integral $\mathfrak{I}$. It should be also noticed, that zeros of $\mathfrak{I}$ and of $\mathrm{Re}(I_m + I_p)$ do not coincide and their shifting is explained by the presence of additional terms in Eq. (6.1).

Such a situation occurs for any intermediate value of the angle ψ. In particular, on considering $\mathrm{Re}(I_m + I_p)$ in the point $x_2 = 0$ under different ψ (see Fig. 7, this function is always positive at the origin) it is seen that this function changes its sign at other critical values (e.g., $\psi' \approx 0.062\pi$, $\psi' < \psi^*$ and $\psi'' \approx 0.241\pi$, $\psi'' > \psi^{**}$ for material (T)). Inside the interval $[\psi', \psi'']$ the value of $\mathrm{Re}(I_m + I_p)$ remains negative and determines the sign changing of the total power flow.

The overall situation looks as follows: at small ψ the density of power flow (sub-integral function in Eq. (3.5)) at $x_2 = 0$ is positive, then at a certain value of angle it equals zero and remains negative for larger ψ. Obviously, for $x_2 \gg 1$ the power flow density is always positive, so an intermediate zone, where the power flow density is negative, exists near the edge. Since the total power flow is obtained by integration for all x_2, its sign will change when this zone is large enough to give the leading contribution into $\mathfrak{I}$. Hence, such zone with a reverse power flow appears at $\psi < \psi^*$ and disappears at $\psi > \psi^{**}$, because for ψ near $\pi/2$ the plate behavior is qualitatively similar to one at small ψ.

7. Leontovich–Lighthill theorem

The results obtained confirm the importance of the adequate formulation of the energy radiation principle for the problems of dynamic bending and do not contradict classical energy relationships. Recalling the known formulation of the energy law in differential form

$$\partial_t e + \operatorname{div} \mathbf{p} = 0, \tag{7.1}$$

where e is the total energy density and $\mathbf{p} = (p_1, p_2), p_\alpha = \mathrm{Re}\,\theta^\bullet_\beta \,\mathrm{Re}\, M_{\alpha\beta} + \mathrm{Re}\, w^\bullet \,\mathrm{Re}\, Q_{\alpha z}$ are components of the Umov–Pointing vector [16]. On varying the frequency and the wave number

$$\omega = \omega_0 + \mathrm{i}\delta\omega, \quad k_1 = k_0 + \mathrm{i}\delta k \tag{7.2}$$

and on introducing the group velocity $V_g = -\delta\omega/\delta k$, after some transformations [17] and after integrating and averaging (3.5), one easily obtains

$$\Im - V_g \tilde{e} = \tfrac{1}{2}\,\mathrm{Re}\{w^{\bullet}\bar{M}_{12}\}\big|_{x_2=0}, \tag{7.3}$$

where

$$\tilde{e} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t \int_0^{+\infty} e \, \mathrm{d}x_2.$$

Thus, $V_g \neq \Im/\tilde{e}$ any longer.

In fact, when averaging over a period and using complex representation for variables such as $A = A^* \mathrm{e}^{\mathrm{i}(\omega t + k_1 x_1 + k_2 x_2)}$ and $B = B^* \mathrm{e}^{\mathrm{i}(\omega t + k_1 x_1 + k_2 x_2)}$ the final contribution into the averaged value of product $\mathrm{Re}(A)\mathrm{Re}(B)$ is given only by $\frac{1}{2}\mathrm{Re}(A\bar{B})$ $(\delta\omega \to 0)$. This leads to the appearance of the exponential term $\exp\{-2(t\delta\omega + x_1\delta k + x_2 \,\mathrm{Im}\, k_2)\}$ in each product and to the replacement for the outer derivatives by $\partial_t \to -2\delta\omega$, $\partial_1 \to -2\delta k$ for components of e and p_1. Since by virtue of the exponential decay at the infinity and of the boundary conditions at the edge

$$p_2 = \mathrm{Re}\,\theta_2^{\bullet}\,\mathrm{Re}\,M_{22} + \mathrm{Re}\,w^{\bullet}\,\mathrm{Re}\,P_{2z} - \partial_1(\mathrm{Re}\,w^{\bullet}\,\mathrm{Re}\,M_{12}),$$

$$\int_0^{+\infty} \partial_2 p_2 \, \mathrm{d}x_2 = p_2|_0^{+\infty} = -p_2|_{x_2=0} = \partial_1(\mathrm{Re}\,w^{\bullet}\,\mathrm{Re}\,M_{12})|_{x_2=0}$$

and from Eq. (7.1) one obtains

$$-2\delta\omega e - 2\delta k p_1 + \partial_2 p_2 + (\cdots) = 0.$$

After integration over x_2 and averaging by time the terms in the brackets disappear and one finally arrives at Eq. (7.3).

As a result the group velocity does not coincide with the ratio of the averaged power flow over the averaged energy density (7.3), which is the essence of *Leontovich–Lighthill theorem* and of the group velocity criterion [16]. Thus, the Leontovich–Lighthill theorem does not hold with the classical formulation for plane waves. The definition of the variable s, which depends only on the stiffness, yields the expression for group velocity

$$V_g = -\frac{\mathrm{d}\omega}{\mathrm{d}k_1} = \frac{2k_1 s^4 d}{\rho V_R},$$

which confirms the sign coincidence of the phase and group velocities. In the meantime it justifies the correctness of the chosen variation (7.2) for the frequency and wave number.

In principle, the situation with a changeable sign of the average power flow is familiar for layered structures. Even some of the classical plane Lamb's waves in isotropic layers with stress-free faces possess such a property [16]. However, this fact remains in accordance with the changeable sign of the group velocity and with the Leontovich–Lighthill theorem. Cases are also known where this theorem is inapplicable. For example, for 3-D-Lamb's waves it does not hold [17], but for the waves with a small front curvature (or at infinity, where in each point the front curvature of the 3-D wave is neglected) it holds for a leading part of the energy and of the average power flow [17]. Hence, the energy radiation principle can be formulated equally for the average power flow and for the group velocity of waves. The case of the plane waves considered in the present paper is different. The sign of group velocity is no longer equal to the sign of power flow due to the formulation of boundary conditions in Kirchhoff's theory of thin plates. And the sign changing of the power flow itself becomes a property of the anisotropy orientation. Notice again that this effect cannot be realized for isotropic materials.

8. Conclusion

One concludes that bending waves of the Rayleigh type in anisotropic plates can be qualitatively different from similar waves in isotropic plates, as well as from the classical Rayleigh waves under plane strains or under plane stresses. These properties are caused by the bending stiffnesses (and especially by the twist coupling stiffnesses) and by the boundary conditions in Kirchhoff's theory of thin plates. Consequently, in the present consideration only integral values of the stifnesses are important and all main effects hold for the plates, made of one or more plies (with symmetrical layup).

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